EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE (2000/2001)

FIRST SEMESTER

MT 207 NUMERICAL ANALYSIS

Answer all questions

Time : Two hours

1. Let $z = \sigma \times (.a_1 a_2 ... a_n a_{n+1} ...)_\beta \times \beta^e$, $\sigma = \pm 1$, $a_1 \neq 0$ be a number in the base β. Express the rounded machine version $fl(z)$ of z, where the number z is rounded to n digits.

   Show that
$$\frac{|z - fl(z)|}{|z|} \leq \frac{1}{2}\beta^{1-n}$$

   Hence deduce that, $fl(z) = (1+\epsilon)z$ with $|\epsilon| \leq \frac{1}{2}\beta^{1-n}$

   Let $S = \sum_{i=1}^{m} x_i$, with $x_1, x_2, ...x_m$ in floating point form. Define S_m by the following recurrence:

   $S_2 = fl(x_1 + x_2)$

   $S_{r+1} = fl(S_r + x_{r+1}), \qquad r = 1, 2, ...m-1$

Show that,

$$| S_m - \sum_{i=1}^{m} | \leq (m-1, m-2, ...2, 1) \begin{pmatrix} | x_1 | \\ | x_2 | \\ .. \\ .. \\ .. \\ | x_m | \end{pmatrix} \frac{1}{2}\beta^{1-n}$$

2. (a) $x = \phi(x)$ is the rearrangement of the equation $f(x) = 0$ and define the iteration,

$$x_{r+1} = \phi(x_r) ; \quad r = 0, 1, 2...... \qquad (1)$$

with the initial value x_0. If $\phi'(x)$ exist, is continuous and $| \phi'(x) | \leq L < 1$ for all x, then show that the sequence $\{x_r\}$ generated by the iteration (1) converges to the unique root α of the equation $f(x) = 0$. Following iterative methods are proposed to compute $(21)^{\frac{1}{3}}$.

Investigate the convergence of the methods.

i. $x_{n+1} = \dfrac{20x_n + \dfrac{21}{(x_n)^2}}{21}$

ii. $x_{n+1} = x_n - \dfrac{(x_n)^4 - 21x_n}{(x_n)^2 - 21}$

iii. $x_{n+1} = (\dfrac{21}{x_n})^{\frac{1}{2}}$

(b) Define the order and the asymptotic error constant of the iteration (1) in part (a).

Show that the order of the Newton's method is 2 and the asymptotic error constant is $\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$. The order of the Secant method is approximately 1.618. Compare the efficiency of the Secant method and Newton's method.

3. (a) Describe the Lagrange's interpolation and, show that the error in the interpolation is given by

$$E(x) = (x - x_0)(x - x_1).....(x - x_n)\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

where $(x_0, x_2, ...x_n)$ are the interpolation points, ξ is in the smallest interval that contains $\{x, x_0, x_1, ...x_n\}$ and $f^{(n+1)}(x) = \dfrac{d^{n+1}f(x)}{dx^{n+1}}$ $f, f^{(1)}, f^{(2)}, ..., f^{(n+1)}$ exist and continuous in $[a, b]$ which contains all $(n+1)$ points and $x \in [a, b]$

(b) Determine the spacing h in a table of equally spaced values of the function $f(x) = \sqrt{x}$ between 1 and 2 so that the interpolation with a second degree polynomial in this table will yield a desired accuracy.

(c) Given that $log_e(1) = 0$, $log_e(2) = 0.69315$, and $log_e(5) = 1.60944$, interpolate with Lagrangian polynomial of the natural logarithms of each integer from 1 to 10.

4. (a) Let $I(f) = \int_a^b f(x)dx$.

$I(p_n)$ is the approximation to $I(f)$ where $p_n(x)$ is the polynomial of degree$\leq n$ which interpolates $f(x)$ at the points $x_0, x_1, \ldots\ldots x_n$, $x_i \in [a, b]$ $\quad i = 0, 1, 2 \ldots. n$, $\quad x_0 \equiv a$ and $x_n \equiv b$.

The error in the approximation is given by,

$$E(f) = I(f) - I(p_n)$$

If $\psi_n(x) = \prod_{j=0}^{n}(x - x_j)$ and $\int_a^b \psi_n(x)dx = 0$,

then $E(f)$ is given by

$$E(f) = \frac{1}{(n+2)!} f^{(n+2)}(\eta) \int_a^b \psi_{n+1}(x)dx,$$

$\eta \in [a, b]$ where $\psi_{n+1}(x) = (x - x_{n+1})\psi_n(x)$
is of one sign on (a, b). $f(x)$ is assumed to be continuously differentiable $(n+2)$ times.

Obtain the Simpson's rule in the form.

$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx = \frac{h}{3}(f_{i-1} + 4f_i + f_{i+1})$$

and the error $E_s = -\frac{1}{90}h^5 f^{(iv)}(\eta_i)$, $\eta_i \in [x_{i-1}, x_{i+1}]$.
Hence obtain the composite Simpson's rule and show that the composite error is

$$-\frac{1}{180}h^4(b-a)f^{(iv)}(\xi), \qquad \xi \in [a, b]$$

(b) Describe Gauss Elimination with scaled partial pivoting. Use the following system to illustrate your answer.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$