EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE 1996/97

(June/July' 2004)

(FIRST SEMESTER)

EXTERNAL DEGREE

EXMT 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

Answer all questions

Time : Three hours

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1. (a) Find the equation of the plane through three given points whose position vectors are $\underline{a}$, $\underline{b}$ & $\underline{c}$ with respect to some origin O.

   Hence or otherwise show that, if the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are such that $\underline{a}$ is perpendicular to both $\underline{b}$ and $\underline{c}$, and $|\underline{b}| = |\underline{c}|$, then the equation of the plane through the three points whose position vectors are $\underline{a}$, $\underline{b}$ and $\underline{c}$ is

$$\left(\frac{\underline{a}}{|\underline{a}|^2} + \frac{\underline{b}+\underline{c}}{|\underline{b}||\underline{c}| + \underline{b}\cdot\underline{c}}\right)\cdot\underline{r} = 1.$$

   (b) If the vector $\underline{x}$ is given by the equation $\lambda\underline{x} + \underline{x}\wedge\underline{a} = \underline{b}$, where $\underline{a}$, $\underline{b}$ are constant vectors and λ is a non-zero scalar, show that

$$\lambda^2(\underline{x}\wedge\underline{a}) + (\underline{a}\cdot\underline{b})\underline{a} - \lambda|\underline{a}|^2\underline{x} + \lambda(\underline{a}\wedge\underline{b}) = 0.$$

   Hence obtain $\underline{x}$ in terms of $\underline{a}$, $\underline{b}$ and λ.

2. (a) Define the following terms.

   i. The gradient of a scalar field ϕ,

   ii. The divergence of a vector field $\underline{F}$,

   iii. The curl of a vector field $\underline{F}$.

   Prove that

$$\text{curl } (\phi \, \underline{F}) = \phi \text{ curl}\underline{F} + \text{grad}\phi \wedge \underline{F} \, .$$

(b) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$. Find $\nabla \left(\frac{1}{r} \right)$.

   If $\underline{a}$ is a constant vector, find

   - $\text{grad}(\underline{a} \cdot \underline{r})$,
   - $\text{curl}(\underline{a} \wedge \underline{r})$.

   Hence show that

   i. $\text{grad}\left(\frac{\underline{a} \cdot \underline{r}}{r^3} \right) = \frac{\underline{a}}{r^3} - \frac{3}{r^5} \, (\underline{a} \cdot \underline{r}) \, \underline{r}$,

   ii. $\text{curl}\left(\frac{\underline{a} \wedge \underline{r}}{r^3} \right) = \frac{2\underline{a}}{r^3} + \frac{3}{r^5} \, (\underline{a} \wedge \underline{r}) \wedge \underline{r}$.

3. (a) State Green's theorem on the plane.

   Verify Green's theorem in the plane for

$$\oint_C (xy + y^2)dx + x^2 dy$$

   where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

(b) State Stoke's theorem.

   Verify the stoke's theorem for a vector $\underline{A} = (2x - y)\underline{i} - yz^2\underline{j} - y^2z\underline{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

4. Prove that the radial and transverse component of the acceleration of a particle in a plane in terms of polar co-ordinates (r, θ) are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

respectively.

A particle of mass m rests on a smooth horizontal table attached through a fixed point on the table by a light elastic string of modulus mg and unstretched length $'a'$. Initially the string is just taut and the particle is projected along the table in a direction perpendicular to the line of the string with velocity $\sqrt{\frac{4ag}{3}}$. Prove that if r is the distance of the particle from the fixed point at time t then

$$\frac{d^2r}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g}{a}(r-a).$$

Prove also that the string will extend until its length is $2a$ and that the velocity of the particle is then half of its initial velocity.

5. A particle moves in a plane with velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the component of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body, where k is a constant. Neglecting the air resistance to the motion of the body, prove that if v is the velocity of the body when its path is inclined an angle ψ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$, the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

6. Establish the equation

$$\underline{F}(t) = m(t)\frac{d\underline{v}}{dt} + \underline{v_0}\frac{dm(t)}{dt}$$

for the motion of a rocket of varying mass $m(t)$ moving in a straight line with velocity $\underline{v}$ under a force $\underline{F}(t)$, matter being emitted at a constant rate with a velocity $\underline{v}_0$ relative to the rocket.

A rocket with initial mass M is fired upwards. Matter is ejected with relative velocity u at a constant rate eM. M' being the mass of the rocket without fuel. Show that the rocket can't rise at once unless $eu > g$.

If it just rises vertically at once, show that it's greatest velocity is

$$u \ln \frac{M}{M'} - \frac{g}{e}\left(1 - \frac{M'}{M}\right)$$

and the greatest height reached is

$$\frac{u^2}{2g}\left\{\ln\left(\frac{M}{M'}\right)\right\}^2 + \frac{u}{e}\left\{1 - \frac{M'}{M} - \ln\left(\frac{M}{M'}\right)\right\}.$$